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m-PRIMARY m-FULL IDEALS

Tae Whan Woo*

Abstract. An ideal I of a local ring (R, m, k) is said to be m-full if there exists an element $x \in m$ such that $Im : x = I$. An ideal I of a local ring R is said to have the Rees property if $\mu(I) > \mu(J)$ for any ideal J containing I. We study properties of m-full ideals and we characterize m-primary m-full ideals in terms of the minimal number of generators of the ideals. In particular, for a m -primary ideal I of a 2-dimensional regular local ring (R, m, k), we will show that the following conditions are equivalent.

1. I is m-full
2. I has the Rees property
3. $\mu(I) = o(I) + 1$

In this paper, let (R, m, k) be a commutative Noetherian local ring with infinite residue field $k = R/m$.

1. Introduction

An ideal I of a local ring (R, m, k) is said to be m-full if there exists an element $x \in m$ such that $Im : x = I$. For example, any prime ideal P of a local ring R is m-full, and depth $R/I > 0$, then I is m-full. Among the source of m-full ideals, more improtant example than any others is integrally closed ideals. In section 2, we show that any integrally closed ideal I of a reduced local domain R is m-full(Corollary 2.9).

To an ideal I of R we associate the following graded rings ;
the associated graded ring of I :

$$G = gr_I(R) = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

the Rees algebra of I :

$$T = R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots$$

the extended Rees algebra of I :

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$$S = R[It, t^{-1}] = \cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus It \oplus I^2t^2 \oplus \cdots$$

where t is a indeterminate over R. Then the integral closure of I is $(t^{-1})\bar{S} \cap R$, where $\bar{S}$ is the integral closure of S.

An ideal I of a local ring R is said to have the Rees property if $\mu(I) \geq \mu(J)$ for any ideal J containing I (Here $\mu(I) = l(I/Im)$ denotes the minimal number of generators of ideal, l stands the length of R-module). Any m-primary m-full ideal of a local ring has the Rees property. (Theorem 2.11) However, if I has the Rees property and Im is m-full, then I is m-full(Theorem 2.13). Also we characterize m-primary m-full ideals in terms of the minimal number of generaters $\mu(I)$ of I and the colength $\Phi(Im)$ of Im. Here, Φ is the map defined by $\Phi(I) = l_{R'}(R'/IR' + YR')$, where R' is the localization of $R[X_1, \cdots, X_d]$ at $mR[X_1, \cdots, X_d]$ and $Y = a_1X_1 + \cdots + a_dX_d$, $m = (a_1, \cdots, a_d)$. In Theorem 2.15, we prove that m-primary ideal I is m-full if and only if $\mu(I) = \Phi(Im)$.

In [6], D.Rees showed that if I is a m-primary integrally closed ideal of a 2-dimensional regular local ring (R, m, k) with infinite residue field $k = R/m$, then $\mu(I) = o(I) + 1$, where $o(I)$ is the integer such that $I \subseteq m^{o(I)}$ and $I \not\subseteq m^{o(I)+1}$. Let I be a m-primary ideal of a 2-dimensional regular local ring (R, m, k). Then the following conditions are equivalent(Corollary 3.3).

1. I is m-full
2. I has the Rees property
3. $\mu(I) = o(I) + 1$

In the latter half of section 3, some good property of m-primary m-full ideal in a 2-dimensional regular local ring has been studied. If I is m-primary m-full, then Im is also m-full.

2. Properties of m-full ideals

Definition 2.1. An ideal I of a local ring R is said to be m-full if there exists an elememt $x \in m - m^2$ such that $Im : x = I$.

Example 2.2.

1. *Let (R, m, k) be a local ring and let P be a prime ideal of R. Then P is a minimal prime ideal containing mP and so P is a prime divisor of mP. Therefore there exists an element $x \in m$ such that $Pm : x = P$.*

2. *Let (R, m, k) be a local ring and let I be m-full ideal of R. Then $I : J$ is m-full for any ideal J of R. Indeed, let $Im : x = I$ and let $y \in (I : J)m : x$. Then $xy \in (I : J)m \subseteq Im : J$ and so $y \in (Im : J) : x = (Im : x) : J = I : J$.*
3. *Let (R, m, k) be a local ring with depth $R/I > 0$. Then there exists an non zero divisor $\bar{x} \in R/I$ and so $I : x = I$. Thus $Im : x = I$.*
4. *Let (R, m, k) be a local ring with depth $R/I > 0$ and let $I = (a_1, a_2, \cdots, a_n)$ be an ideal generated by a regular sequence. Then I^r is m-full for any integer $r \geq 1$. We prove by induction on r. Since depth $R/I > 0$, there exists $x \in m$ such that $I = I : x$. Suppose that $I^{r-1}m : x = I^{r-1}$. If $y \in I^r m : x$, then $xy \in I^r m \subseteq I^{r-1}m$ and so $y \in I^{r-1}$ by the inductive hypothesis. Hence we can write $y = F(a_1, a_2, \cdots, a_n)$ with $F(X_1, X_2, \cdots, X_n) \in R[X_1, X_2, \cdots, X_n]$ homogeneous degree $r - 1$. Since*

$$xy = xF(a_1, a_2, \cdots, a_n) \in I^r m \subseteq I^r$$

   and $a_1, a_2, \cdots, a_n$ is a regular sequence, every coefficient of $xF(X_1, X_2, \cdots, X_n)$ belongs to $I = I : x$. So every coefficient of $F(X_1, X_2, \cdots, X_n)$ belongs to I. Therefore $y \in I^r$.
5. *Let X be an indeterminate over a local ring R with depth $R/I > 0$. Then $I\,[|X|]$ is m'-full in $R\,[|X|]$ where m' is the maximal ideal of $R\,[|X|]$.*

Another important example of m-full ideals is integrally closed ideals. Let I be an ideal of a Noetherian ring R. An element $x \in R$ is said to be integral over I if $x^n + a_1 x^{n-1} + \cdots + a_n = 0,\ a_i \in I^i$. The set of all elements of R which are integral over I is called the integral closure of I, and denoted by $\bar{I}$. An ideal I is said to be integrally closed if $\bar{I} = I$. The next lemma is well known.

Lemma 2.3. *Let I be an ideal of a Noetherian ring R and let $R[It, t^{-1}]$ be the extended Rees algebra of I, $R[It, t^{-1}] = \cdots \oplus Rt^{-1} \oplus R \oplus It \oplus I^2t^2 \oplus \cdots$. Then an element $x \in R$ is integral over I if and only if $xt \in R[t, t^{-1}]$ is integral over $R[It, t^{-1}]$. (Here t is an indeterminate over R)*

Corollary 2.4. *Let I be an ideal of a Noetherian ring R. Then $\bar{I}$ is an ideal of R.*

Proof. Let x and y be elements in $\bar{I}$ and let $r \in R$. Then xt and yt in $R[t, t^{-1}]$ are integral over $R[It, t^{-1}]$ by Lemma 2.3. Thus $xt + yt = (x + y)t$ and rxt are integral over $R[It, t^{-1}]$. From Lemma 2.3, we have $x + y$ and rx are in $\bar{I}$. □

Corollary 2.5. *Let I be an ideal of Noetherian ring R and let $\bar{S}$ be the integral closure of $S = R[It, t^{-1}]$ in its total quotient ring. Then $\bar{I} = (t^{-1})\bar{S} \cap R$.*

Proposition 2.6. *Let I be an ideal of a Noetherian local ring R. If the associated graded ring of I is reduced, then I^r is integrally closed for any $r \geq 0$.*

Proof. We use induction on r. Suppose $r \geq 1$ and I^{r-1} is integrally closed. Let $x \in \bar{I^r}$. Then x satisfies

$$x^n + a_1x^{n-1} + \cdots + a_n = 0, \quad a_i \in I^{ri}.$$

Thus

$$x^n = -(a_1x^{n-1} + \cdots + a_n) \in I^{nr-n+1}$$

since each $a_ix^{n-i} \in I^{ri+(n-i)(r-1)} \subseteq I^{nr-n+1}$. Let $\bar{x}$ be the image of x in $I^{r-1}/I^r \subseteq gr_I(R)$. Then $(\bar{x})^n \in I^{n(r-1)}/I^{nr-n+1}$. Hence $(\bar{x})^n = 0$. So, $\bar{x} = 0$ since $gr_I(R)$ is reduced. Therefore $x \in I^r$. □

Corollary 2.7. *Let (R, m, k) be a regular local ring. Then m^r is integrally closed for any $r \geq 1$.*

Proof. It is enough to note that $gr_m(R)$ is the polynomial ring $k[X_1, \cdots, X_d]$ which is a domain where $X_1, \cdots, X_d$ are indeterminates over k. □

It is shown that the integral closure of a Noetherian ring is a Krull ring. Since $R[It, t^{-1}]$ is a Noetherian ring, $\bar{S}$ is a Krull ring with the same notation as in Corollary 2.5. Now we prove that any integrally closed ideal is m-full.

Theorem 2.8. *Let (R, m, k) be a reduced local domain and let I be an ideal of R. Then there exists an element $x \in m$ such that $I \subseteq Im : x \subseteq \bar{I}$.*

Proof. Put $S = R[It, t^{-1}]$. Then, since $\bar{S}$ is a Krull ring, we have a primary decomposition

$$(t^{-1})\bar{S} = q_1 \cap \cdots \cap q_r$$

of $(t^{-1})\bar{S}$ with each $P_i = \sqrt{q_i}$ a height 1 prime ideal of $\bar{S}$. For each $i = 1, 2, \cdots r$, we consider the discrete valuation ring $V_i = \bar{S_{P_i}}$. Since $k = R/m$ is infinite, there exists an element $x \in m - \bigcup_{i=1}^{r}(P_iV_imV_i \cap R)$ such that $mV_i = xV_i$ for all $1 \leq i \leq r$.

Now let $y \in Im : x$. Then as $\dfrac{xy}{1} \in mV_iIV_i = xV_iIV_i$, we have $\dfrac{y}{1} \in IV_i \subseteq q_iV_i$ for all $1 \leq i \leq r$. Hence $y \in (\bigcap_{i=1}^{r} q_i) \cap R = \bar{I}$ since $\bar{I} = (t^{-1})\bar{S} \cap R$ by Corollary 2.5. □

COROLLARY 2.9. *Any integrally closed ideal of a reduced local domain (R, m, k) is m-full.*

Now, in the remainder of this paper, we will consider mostly m-primary m-full ideals. Actually, m-primary m-full ideals have good properties than that of non m-primary ideals. Also, m-primary m-full ideals are characterized in terms of the minimal number of generators of ideal and the colength.

DEFINITION 2.10. Let (R, m, k) be a local ring. An ideal I of R is said to have the Rees property if $\mu(I) \geq \mu(J)$ for any ideal J containing I. (Here $\mu(I) = l(I/Im)$ denotes the minimal number of generators of ideal, l stands the length of R-module)

THEOREM 2.11. *Let (R, m, k) be a local ring. Then any m-primary m-full ideal has the Rees property.*

Proof. Let $Im : x = I$. By the exact sequence

$$0 \to R/Im : x \to R/Im \to R/Im + xR \to 0$$

(Here $\mu_x : R/Im : x \to R/Im$ is the map defined by $\mu_x(r + Im : x) = xr + Im$), we have

$$l(R/Im + xR) = l(Im : x/Im) = l(I/Im) = \mu(I).$$

Hence for any ideal J containing I

$$\begin{aligned} \mu(J) &= l(J/Jm) \\ &\leq l(Jm : x/Jm) = l(R/Jm + xR) \\ &\leq l(R/Im + xR) = \mu(I). \end{aligned}$$

Thus I has the Rees property. □

REMARK 2.12. If a parameter ideal I of a d-dimensional local ring R is m-full, then R is a regular local ring. Indeed, since I has the Rees property,

$$d = \dim R \leq \mu(m) \leq \mu(I) = d.$$

Hence R is a regular local ring.

Proposition 2.13. *Let (R, m, k) be a local ring and let I be an ideal of R. If $Im : x = I$, then $I : m = I : x$.*

Proof. Since $Im : x = I$,

$$I : x \subseteq (Im : m) : x = (Im : x) : m = I : m.$$

Hence $I : x = I : m$. □

Theorem 2.14. *Let I be an ideal of R such that Im is m-full. If I has the Rees property, then I is m-full.*

Proof. We will show that $Im : m = I$. Suppose $I \subseteq Im : m$ and let $J = Im : m$. Then $\mu(I) \geq \mu(J)$ since I has the Rees property. On the other hand, $Jm = Im$ and so

$$\mu(J) = l(J/Jm) = l(J/Im) > l(I/Im) = \mu(I).$$

This is a contradiction and we must have $Im : m = I$. Since Im is m-full, there exists an element $x \in m$ such that $Im : x = Im : m$. Therefore $I = Im : m = Im : x$. □

Definition 2.15. Let (R, m, k) be a local ring with the maximal ideal $m = (a_1, \cdots, a_d)$ and let I be a m-primary ideal of R. Let R' denotes $R[X_1, \cdots, X_d]$ localized at $mR[X_1, \cdots, X_d]$, where $X_1, \cdots, X_d$ are indeterminates over R. Put $Y = a_1X_1 + \cdots + a_dX_d$. For any m-primary ideal I, we define the colength of I by $\Phi(I) = l_{R'}(R'/IR' + YR')$. In particular, an element $x \in m - m^2$ is called a general element for I if $\Phi(I) = l_R(R/I + xR)$.

Remark 2.16.

1. In general, $\Phi(I) \leq l_R(R/I + xR)$ for any $x \in m$. But for any m-primary ideal I, a general element exists always.
2. Let I be a m-primary ideal. From the exact sequence

$$0 \to I \to I : m \to I : m/I \to 0$$

we have the exact sequence

$$0 \to I \otimes R' \to (I : m) \otimes R' \to (I : m/I) \otimes R' \to 0.$$

So $(I : m/I) \otimes R' \cong (I : m)R'/IR'$. Note that $(I : m)R' = IR' : mR'$. Hence

$$\begin{aligned} l(IR' : mR'/I') &= l((I : m)R'/IR') \\ &= l((I : m/I) \otimes R') \\ &= l(I : m/I). \end{aligned} \tag{2.1}$$

THEOREM 2.17. *Let (R, m, k) be a local ring and let I be a m-primary ideal. Then the following conditions are equivalent.*

1. *I is m-full*
2. *$\mu(I) = \Phi(Im)$*

Proof. $1 \Rightarrow 2$; Let $Im : x = I$. Then $Im : m = Im : x$ since $Im : x = I \subseteq Im : m$. From the exact sequence

$$0 \to R/Im : x \to R/Im \to R/Im + xR \to 0$$

we have

$$\begin{aligned} l(R/Im + xR) &= l(Im : x/Im) \\ &= l(Im : m/Im) \\ &= l(ImR' : mR'/ImR'). \end{aligned} \tag{2.2}$$

Also from the exact sequence

$$0 \to R'/ImR' : Y \to R'/ImR' \to R'/ImR' + YR' \to 0$$

we have

$$\begin{aligned} \Phi(Im) &= l(R'/ImR' + YR') \\ &= l(ImR' : Y/ImR') \\ &\geq l(ImR' : mR'/ImR') \\ &= l(R/Im + xR). \end{aligned} \tag{2.3}$$

Hence $\Phi(Im) = l(R/Im + xR)$ by the above Remark 2.16 (Note that x is a general element for Im). On the other hand, from the first exact sequence, we have

$$\mu(I) = l(I/Im) = l(Im : x/Im) = l(R/Im + xR). \tag{2.4}$$

Hence $\mu(I) = \Phi(Im)$.

$2 \Rightarrow 1$; Let x be a general element for Im. Then $\Phi(Im) = l(R/Im + xR)$ and so

$$l(Im : x/Im) = l(R/Im + xR) = \Phi(Im) = \mu(I) = l(I/Im). \tag{2.5}$$

Hence $Im : x = I$. $\square$

From the proof of Theorem 15, we have easy consequences.

COROLLARY 2.18. *Let (R, m, k) be a local ring and let I be a m-primary ideal of R. If $Im : x = I$, then x is a general element for Im. Also if I is m-full, then $Im : x = I$ for any general element x in Im.*

Proof. Let x be any general element for Im. Then,

$$\mu(I) = \Phi(Im) = l(R/Im + xR) = l(Im : x/Im).$$

Hence $Im : x = I$ for any general element in Im. □

3. m-full ideals in a 2-dimensional regular local ring

Now, we assume that (R, m, k) is a 2-dimensional regular local ring with infinite residue field $k = R/m$ and let I be a m-primary ideal of R. In Theorem 2.11, we prove that any m-primary m-full ideal has the Rees property.

LEMMA 3.1. *Let I be a m-primary ideal of R. Then there exists an element $x \in m - m^2$ such that $l(R/I + xR) = o(I)$.*

Proof. Let $o(I) = n$ and let $m = (t_1, t_2)$. Then I contains an element a such that $a \in m^n - m^{n+1}$. Write

$$a = b_0 t_1^n + b_1 t_1^{n-1} t_2 + \cdots + b_n t_2^n, \quad b_i \in R.$$

Take a linear transformation $t_1 \to t_1$, $t_2 \to z t_1 + t_2$ for some $z \in R - m$. Then

$$a = c_0 t_1^n + c_1 t_1^{n-1} t_2 + \cdots,$$

where c_0 is a unit. Let $x = t_2$. Then R/xR is a discrete valuation ring and $I \equiv t_1^n R \,(\operatorname{mod} xR)$. Therefore $l(R/I + xR) = n$. □

THEOREM 3.2. *Let I be a m-primary ideal of R. If I has the Rees property, then I is m-full.*

Proof. Let $o(I) = n$. Then there exists an element $x \in m - m^2$ such that $l(R/Im + xR) = n + 1$ by Lemma 3.1. Therefore

$$l(R/Im + xR) = n + 1 = \mu(m^n) \leq \mu(I) = l(I/Im) \tag{3.1}$$

since I has the Rees property. Now from the exact sequence

$$0 \to R/Im : x \to R/Im \to R/Im + xR \to 0$$

we have

$$l(Im : x/Im) = l(R/Im + xR) \leq l(I/Im).$$

Hence $Im : x = I$ and so I is m-full. □

THEOREM 3.3. *Let I be a m-primary ideal of R. Then the following conditions are equivalent.*

1. *I is m-full*
2. *I has the Rees property*

3. $\mu(I) = o(I) + 1$

Proof. $1 \Rightarrow 2$; Theorem 2.11.
$2 \Rightarrow 3$; Since I has the Rees property, $\mu(m^{o(I)}) \leq \mu(I)$. Note that there exists an element $x \in m - m^2$ such that $l(R/Im + xR) = o(Im) = o(I) + 1$ by Lemma 3.1. Hence

$$\begin{aligned} l(Im : x/Im) &= l(R/Im + xR) = o(I) + 1 \\ &= \mu(m^{o(I)}) \leq \mu(I) \\ &= l(I/Im) \leq l(Im : x/Im). \end{aligned} \tag{3.2}$$

Thus $\mu(I) = o(I) + 1$.
$3 \Rightarrow 1$; Let x be an element in m such that $l(R/Im + xR) = o(I) + 1$. Then

$$l(Im : x/Im) = l(R/Im + xR) = o(I) + 1 = \mu(I) = l(I/Im). \tag{3.3}$$

Hence $Im : x = I$. □

Proposition 3.4. *Let I be a m-primary ideal of R. If $Im : x = I$, then $l(R/I + xR) = o(I)$.*

Proof. In general $l(R/I + xR) \geq o(I)$ since

$$m + xR \supseteq m^2 + xR \supseteq \cdots \supseteq m^{o(I)} + xR \supseteq I + xR$$

for any $x \in m$. Suppose $l(R/I + xR) > o(I)$. Then from the exact sequence

$$0 \to R/Im : x \to R/Im \to R/Im + xR \to 0$$

we have $l(R/Im + xR) = \mu(I)$. Since I is m-full,

$$o(I) + 1 = \mu(I) = l(R/m + xR) \geq l(R/I + xR) > o(I). \tag{3.4}$$

So $l(R/Im + xR) = l(R/I + xR)$. From the fact

$$l(R/Im + xR) = l(R/I + xR) + (I + xR/xR)$$

we have $\mu(I + xR/xR) = 0$. Thus $I \subseteq xR$. Let $m = (x, y)$ and let $a = ry \notin xR$ such that $rx \in I$. Then $a \notin I$ and $ax = ryx \in Im$, so $a \in Im : x$. This is a contradiction. □

Remark 3.5. If $Im : x = I$, then $Im : m = Im : x$. Thus $l(R/Im + xR) = o(Im) = o(I) + 1$.

Theorem 3.6. *A primary ideal I of R is m-full if and only if $I : m = I : x$ for some $x \in m - m^2$ such that $l(R/I + xR) = o(I)$.*

Proof. Suppose I is m-full. Then $Im : x = I$ for some $x \in m$ such that $l(R/I + xR) = o(I)$ by proposition 3.4. Hence $I : m = I : x$ for some $x \in m$ such that $l(R/I + xR) = o(I)$.

Conversely, let $o(I) = r$. From the exact sequence

$$0 \to R/I : x \to R/I \to R/I + xR \to 0$$

we have

$$\begin{aligned} r &= l(R/I + xR) = l(I : x/I) \\ &= l(I : m/I) = dim_k(I : m/I). \end{aligned} \tag{3.5}$$

On the other hand, from the resolution of $k = R/m$, we have

$$\operatorname{Tor}_2^k(R/I, k) \cong I : m/I.$$

Now applying the Hilbert-Burch Theorem for the resolution of R/I, we know that $\operatorname{Tor}_2^k(R/I, k)$ is a $\mu(I) - 1$ dimensional k-vector space. Hence $r = \mu(I) - 1$ and I is m-full by Corollary 3.3. □

LEMMA 3.7. *Let I and J be m-primary ideals of R. If $I : m = I : x$ and $J : m = J : x$, then $IJ : m = IJ : x$.*

Proof. Put $m = (x, y)$. Since $R/(x)$ is a discrete valuation ring, there exist $x_1 \in I$, $x_2 \in J$ such that

$$(I, x) = (x_1, x), \quad (J, x) = (x_2, x).$$

Note that neither x_1 nor x_2 is divisible by x. Let $\alpha \in IJ : x$. Then

$$\alpha x = \sum(\alpha_{1j}x_1 + \beta_{1j}x)(\alpha_{2j}x_2 + \beta_{2j}x), \quad \alpha_{1j},\ \beta_{1j},\ \alpha_{2j},\ \beta_{2j}\ \in R$$

since $\beta_{1j}y \in I$, $\beta_{2j}y \in J$ as for $\beta_{1j} \in I : x = I : m$, $\beta_{2j} \in J : x = J : m$.

On the other hand, since $\sum \alpha_{1j}\alpha_{2j}x_1x_2$ is divisible by x, it follows that $\sum \alpha_{1j}\alpha_{2j}$ is divisible by x. Hence

$$\alpha xy = (\beta x_1 x_2 + \gamma)x, \quad \beta \in R, \quad \gamma \in IJ.$$

Thus $\alpha y \in IJ$ and so $\alpha m \subseteq IJ$. □

THEOREM 3.8. *Let (R, m, k) be a 2-dimensional regular local ring. If I is a m-primary m-full, then Im is also m-full.*

Proof. Let $Im : x = I$. Then $I : m = I : x$ and $m : x = m : m$. Hence $Im : x = Im : m$ by Lemma 3.7. But $l(R/Im + xR) = o(Im)$ since I is m-full, so Im is m-full by Theorem 3.5.

□

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School of Liberal Arts
Seoul National University of Science and Technology
Seoul 139-743, Republic of Korea
E-mail: thwoo@seoultech.ac.kr